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SOLVING SOME FIRST ORDER DIFFERENTIAL RECURRENCE EQUATIONS WITH DISCRETE AUTO-CONVOLUTION

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Abstract

A differential recurrence equation consists of a recurrent sequence of differential equations, from which a sequence of unknown functions must be determined. In this paper, we will present several methods for solving two nonlinear (quadratic) first-order homogeneous differential recurrence equations with discrete auto-convolution of the unknown functions or their derivatives. We use here three types of proofs: The first by mathematical induction, the second based on generating function method, and the third by a substitution which reduces the differential recurrence equation to the corresponding algebraic recurrence equation. We will present these methods on the simplest differential recurrence equations with discrete auto-convolution. For the first equation, we will determine the solutions that are in geometric progression, while the second is solved without any supplementary condition. Finally, we present two differential recurrence equations with combinatorial auto-convolution that are reduced to the first ones by substitutions, and some applications of the results from this paper to the discrete linear time-invariant physical systems theory are also presented.

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1. Introduction

Differential recurrence equations were considered, for example, in the papers [8]-[10] and in the author's books and papers [1]-[6]. In the paper [3] and [5], see also the book [1], the author considered a new type of such equations, namely, differential recurrence equations with discrete auto-convolution. These results were used in [4] to solve an integral recurrence equation with auto-convolution, which was then solved in the paper [6] in a different manner, namely, using the hybrid Laplace transformation. Here the simplest two dual first-order differential recurrence equations with auto-convolution are considered and their solutions are obtained. While the second equation will be considered here for the first time, the first is a particular case of a more general equation, considered in [5]. For completeness, this equation will be treated in detail, as the second, because the actual treatment will contain many changes which give simplifications and clarifications compared to the general theory given in [5].

2. The Equations

Being given two sequences of functions $x(t) = (x_0(t), x_1(t), \ldots, x_n(t), \ldots)$ and $y(t) = (y_0(t), y_1(t), \ldots, y_n(t), \ldots)$, one calls *discrete convolution* or *Cauchy product* of them (see, for example, [1] and [2]), the sequence of functions

$$x(t) * y(t) = \left(x_0(t)y_0(t),\ x_0(t)y_1(t) + x_1(t)y_0(t),\ \ldots,\ \sum_{k=0}^{n} x_k(t)y_{n-k}(t), \ldots\right).$$

Particularly, if $x(t) = y(t)$, the product is called the *auto-convolution* of $x(t)$ and it is denoted

$$x^{2*}(t) = \underbrace{x(t) * x(t)}_{2} = \left(x_0^2(t),\ 2x_0(t)x_1(t),\ \ldots,\ \sum_{k=0}^{n} x_k(t)x_{n-k}(t), \ldots\right).$$

Also, if the terms of the sequence $x(t) = (x_n(t))$ are differentiable functions, condition which will be supposed in this paper, one considers its *derivative* $x'(t) = (x'_n(t))$. In the following, we present the main methods to solve the simple “dual” differential recurrence equations with discrete auto-convolution $x'(t) = x^{2*}(t)$ and $x(t) = (x'(t))^{2*}$. For the first equation, we determine the solutions that form a geometrical progression, while for the second no condition will be imposed.

3. A Recurrence Differential Equation with Discrete Auto-Convolution

Theorem 1. *The sequence of solutions that form a geometric progression of the equation*

$$x'_n(t) = \sum_{k=0}^{n} x_k(t) x_{n-k}(t), \quad n = 0, 1, 2, \ldots, \tag{1}$$

is given by the formula

$$x_n(t) = \frac{(-1)^{n+1} C_1^n}{(t + C_0)^{n+1}}, \quad n = 0, 1, 2, \ldots, \tag{2}$$

where C_0 *and* C_1 *are arbitrary constants.*

3.1. Proof of Theorem 1 by mathematical induction

For $n = 0$, the Equation (1) is reduced to $x'_0(t) = x_0^2(t)$, hence $\frac{x'_0(t)}{x_0^2(t)} = 1$, with the solution $x_0(t) = -\frac{1}{t + C_0}$ and for $n = 1$ to $x'_1(t) = 2x_0(t)x_1(t)$, hence $\frac{x'_1(t)}{x_1(t)} = -\frac{2}{t + C_0}$, with the solution $x_1(t) = \frac{C_1}{(t + C_0)^2}$.

For $n \geq 2$, the Equation (1) is reduced to $x'_n(t) = 2x_0(t)x_n(t) + \sum_{k=1}^{n-1} x_k(t) x_{n-k}(t)$, hence to $x'_n(t) + \frac{2}{t + C_0} x_n(t) = \sum_{k=1}^{n-1} x_k(t) x_{n-k}(t)$, with the solution

$$x_n(t) = e^{-\int \frac{2dt}{t+C_0}}\left[\int e^{\int \frac{2dt}{t+C_0}} \sum_{k=1}^{n-1} x_k(t)x_{n-k}(t)dt + C_n\right]$$

$$= \frac{1}{(t+C_0)^2}\left[\int (t+C_0)^2 \sum_{k=1}^{n-1} x_k(t)x_{n-k}(t)dt + C_n\right].$$

For $n = 2$, the solution becomes

$$x_2(t) = \frac{1}{(t+C_0)^2}\left[\int (t+C_0)^2 x_1^2(t)dt + C_2\right]$$

$$= \frac{1}{(t+C_0)^2}\left[\int \frac{C_1^2}{(t+C_0)^2}dt + C_2\right]$$

$$= -\frac{C_1^2}{(t+C_0)^3} + \frac{C_2}{(t+C_0)^2}.$$

Because the sequence $x_n(t)$ is a geometric progression, we have $x_1^2(t) = x_0(t)x_2(t)$, hence

$$\frac{C_1^2}{(t+C_0)^4} = -\frac{1}{t+C_0}\left[-\frac{C_1^2}{(t+C_0)^3} + \frac{C_2}{(t+C_0)^2}\right],$$

from which one obtains $C_2 = 0$. Therefore, $x_2(t) = -\frac{C_1^2}{(t+C_0)^3}$.

We suppose that $x_k(t) = \frac{(-1)^{k+1}C_1^k}{(t+C_0)^{k+1}}$ and $C_k = 0$, for $2 \le k \le n-1$.

Then we have

$$x_n(t) = \frac{1}{(t+C_0)^2}\left[\int (t+C_0)^2 \sum_{k=1}^{n-1} \frac{(-1)^{k+1}C_1^k}{(t+C_0)^{k+1}} \frac{(-1)^{n-k+1}C_1^{n-k}}{(t+C_0)^{n-k+1}}dt + C_n\right]$$

$$= \frac{1}{(t+C_0)^2}\left[\int (t+C_0)^2 \sum_{k=1}^{n-1} \frac{(-1)^n C_1^n}{(t+C_0)^{n+2}}dt + C_n\right]$$

$$= \frac{1}{(t+C_0)^2}\left[\int \frac{(n-1)(-1)^n C_1^n}{(t+C_0)^n}\,dt + C_n\right] = \frac{(-1)^{n+1}C_1^n}{(t+C_0)^{n+1}} + \frac{C_n}{(t+C_0)^2}.$$

Because the sequence $x_n(t)$ is a geometric progression, we have $x_{n-1}^2(t) = x_{n-2}(t)x_n(t)$, hence $\frac{C_1^{2n-2}}{(t+C_0)^{2n}} = \frac{(-1)^{n-1}C_1^{n-2}}{(t+C_0)^{n-1}}\left[\frac{(-1)^{n+1}C_1^n}{(t+C_0)^{n+1}} + \frac{C_n}{(t+C_0)^2}\right]$, from which one obtains $C_n = 0$. Therefore, $x_n(t) = \frac{(-1)^{n+1}C_1^n}{(t+C_0)^{n+1}}$.

According to the mathematical induction axiom, the solution $x_n(t)$ is given by the formula (2).

3.2. Proof of Theorem 1 by generating function method

We consider the generating function $G(t, z) = \sum_{n=0}^{\infty} x_n(t)z^n$ of the sequence of functions $x_n(t)$, defined by a formal series. Multiplying Equation (1) with z^n and summing, it follows that

$$\sum_{n=0}^{\infty} x_n'(t)z^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} x_k(t)x_{n-k}(t)z^n.$$

Considering the formula of the product of the power series, one obtains the differential equation $\frac{\partial}{\partial t}G(t, z) = G^2(t, z)$, with the solution $G(t, z) = -\frac{1}{t + C(z)}$, where $C(z) = \sum_{n=0}^{\infty} C_n z^n$ is a constant in t. Therefore, one obtains $[t + C(z)]G(t, z) = -1$, hence

$$\left[t + C_0 + \sum_{n=1}^{\infty} C_n z^n\right]\sum_{n=0}^{\infty} x_n(t)z^n = -1.$$

Using the formula of the product of the power series and identifying the coefficients, it results that

$$(t + C_0)x_0(t) = -1 \text{ and } (t + C_0)x_n(t) + \sum_{k=1}^{n} C_k x_{n-k}(t) = 0, \quad \forall n \geq 2. \quad (3)$$

From the first relation, one obtains $x_0(t) = -\frac{1}{t + C_0}$. From (3), for $n = 1$, it follows that $x_1(t) = \frac{C_1}{(t + C_0)^2}$ and for $n = 2$, that $x_2(t) = -\frac{C_1^2}{(t + C_0)^3} + \frac{C_2}{(t + C_0)^2}$. Because the sequence $x_n(t)$ is a geometric progression, as in the first proof one obtains $C_2 = 0$, hence $x_2(t) = -\frac{C_1^2}{(t + C_0)^3}$. Using the mathematical induction axiom, from the relation (3), it follows as in the above proof that $C_n = 0$, $\forall n \geq 2$ and $x_n(t)$ has the form (2).

Remark. Alternatively, this method can be presented as the Z transformation method, where $Z(x_n(t))(z) = \sum_{n=0}^{\infty} x_n(t) z^{-n} = G\left(t, \frac{1}{z}\right)$. The same remark can also be made for the proof based on the generating function given below in Theorem 2.

3.3. Proof of Theorem 1 by substitution

We will use in the Equation (1) the substitution

$$x_n(t) = \frac{a_n}{x^{n+1}(t)}, \quad n = 0, 1, 2, \ldots, \quad (4)$$

where $x(t)$ is a differentiable function and a_n are real numbers, all of which should be determined. Then the Equation (1) becomes successively

$$-\frac{(n+1)a_n}{x^{n+2}(t)} = \sum_{k=0}^{n} \frac{a_k}{x^{k+1}(t)} \frac{a_{n-k}}{x^{n-k+1}(t)}, \quad -\frac{(n+1)a_n}{x^{n+2}(t)} = \sum_{k=0}^{n} \frac{a_k a_{n-k}}{x^{n+2}(t)},$$

$$-(n+1)a_n = \sum_{k=0}^{n} a_k a_{n-k}.$$

For $n = 0$, one obtains the nonzero number $a_0 = -1$. Denoting $b_n = -a_n$, the Equation (1) is reduced to the algebraic recurrence equation $(n+1)b_n = \sum_{k=0}^{n} b_k b_{n-k}$, $n = 0, 1, 2, \ldots$, and $b_0 = 1$. On the other hand, for $n = 0$, it follows from (1) and (4) that $x_0(t) = -\frac{1}{t + C_0} = \frac{a_0}{x(t)} = -\frac{1}{x(t)}$, hence $x(t) = t + C_0$, where $C_0 = x(0)$. From the lemma below, it follows that $b_n = b_1^n$, hence $a_n = (-1)^{n+1} a_1^n$ and from (4) one obtains the formula (2), where $a_1 = C_1$.

Lemma 3.4. *The algebraic recurrence equation*

$$(n+1)b_n = \sum_{k=0}^{n} b_k b_{n-k}, \quad n = 0, 1, 2, \ldots, \tag{5}$$

has the solution

$$b_n = b_1^n, \quad n = 0, 1, 2, \ldots. \tag{6}$$

3.5. Proof of lemma by mathematical induction

For $n = 0$, the equation becomes $b_0 = b_0^2$ and has the nonzero solution $b_0 = 1$. For $n = 1$, the equation is obvious and for $n = 2$, it becomes $3b_2 = 2b_0 b_2 + b_1^2$ and has the solution $b_2 = b_1^2$. We suppose that $b_k = b_1^k$, $k \leq n-1$. Then the Equation (5) gives $(n+1)b_n = 2b_0 b_n + \sum_{k=1}^{n-1} b_k b_{n-k} = 2b_n + \sum_{k=1}^{n-1} b_1^k b_1^{n-k} = 2b_n + \sum_{k=1}^{n-1} b_1^n = 2b_n + (n-1)b_1^n$, hence $b_n = b_1^n$, $n = 0, 1, 2, \ldots$.

3.6. Proof of lemma by generating function method

We consider the generating function $G(z) = \sum_{n=0}^{\infty} b_n z^n$ of the numerical sequence (b_n), defined by a formal series. By multiplying Equation (1) with z^n and summing, we get $\sum_{n=0}^{\infty}(n+1)b_n z^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} b_k b_{n-k} z^n$. Considering the formula of the power series multiplication, one obtains the differential equation $[zG(z)]' = G^2(z)$, which successively takes the forms $zG'(z) + G(z) = G^2(z)$, $\frac{G'(z)}{G(z)[G(z)-1]} = \frac{1}{z}$, and $\frac{G'(z)}{G(z)-1} - \frac{G'(z)}{G(z)} = \frac{1}{z}$. By integration, it follows successively that $\ln\left|\frac{G(z)-1}{G(z)}\right| = \ln|Cz|$ and $\frac{G(z)-1}{G(z)} = Cz$, from which one obtains the solution $G(z) = \frac{1}{1-Cz} = \sum_{n=0}^{\infty} C^n z^n$, where C is an arbitrary constant. By identifying the coefficients in the two expressions of $G(z)$, one obtains $b_n = C^n$. For $n = 1$, we obtain $C = b_1$, therefore the solutions b_n of the algebraic recurrence equation (5) are given by the formula (6).

3.7. Reciprocal proof for lemma

If $b_n = b_1^n$, we have $\sum_{k=0}^{n} b_k b_{n-k} = \sum_{k=0}^{n} b_1^k b_1^{n-k} = \sum_{k=0}^{n} b_1^n = (n+1)b_1^n = (n+1)b_n$, $n = 0, 1, 2, \ldots$, therefore these numbers b_n satisfy the Equation (5).

3.8. Reciprocal proof for Theorem 1

If $x_n(t) = \frac{(-1)^{n+1}C_1^n}{(t+C_0)^{n+1}}$, then we have $\sum_{k=0}^{n} x_k(t)x_{n-k}(t) = \sum_{k=0}^{n} \frac{(-1)^{k+1}C_1^k}{(t+C_0)^{k+1}} \frac{(-1)^{n-k+1}C_1^{n-k}}{(t+C_0)^{n-k+1}} = \sum_{k=0}^{n} \frac{(-1)^n C_1^n}{(t+C_0)^{n+2}} = (n+1)\frac{(-1)^n C_1^n}{(t+C_0)^{n+2}} = x_n'(t)$, $n = 0, 1, 2, \ldots$, therefore the functions $x_n(t)$ satisfy the Equation (1).

4. The Dual Equation

Theorem 2. *The solutions of the equation*

$$x_n(t) = \sum_{k=0}^{n} x'_k(t)x'_{n-k}(t), \quad n = 0, 1, 2, \ldots, \tag{7}$$

are given by the formulas

$$x_0(t) = \frac{1}{4}(t + C_0)^2, \quad x_1(t) = C_1(t + C_0), \tag{8}$$

$$x_n(t) = C_n(t + C_0) + \sum_{k=1}^{n-1} C_k C_{n-k}, \quad n = 2, 3, \ldots, \tag{9}$$

where C_n *are arbitrary constants.*

4.1. Proof of Theorem 2 by mathematical induction

For $n = 0$, the Equation (7) is reduced to $x_0(t) = [x'_0(t)]^2$, hence $\frac{x'_0(t)}{\sqrt{x_0(t)}} = 1$. By integration, we obtain $2\sqrt{x_0(t)} = t + C_0$, hence it follows that the solution is $x_0(t) = \frac{1}{4}(t + C_0)^2$. For $n = 1$, the Equation (7) becomes $x_1(t) = 2x'_0(t)x'_1(t)$, hence $\frac{x'_1(t)}{x_1(t)} = \frac{1}{t + C_0}$, with the solution $x_1(t) = C_1(t + C_0)$, therefore the solutions $x_0(t)$ and $x_1(t)$ are given by formula (8). For $n \geq 2$, the Equation (7) becomes $x_n(t) = 2x'_0(t)x'_n(t) + \sum_{k=1}^{n-1} x'_k(t)x'_{n-k}(t)$, hence $x'_n(t) - \frac{1}{t + C_0}x_n(t) = -\frac{1}{t + C_0}\sum_{k=1}^{n-1} x'_k(t)x'_{n-k}(t)$, with the solution

$$x_n(t) = e^{\int \frac{dt}{t+C_0}}\left[-\int \frac{1}{t + C_0}e^{-\frac{dt}{t+C_0}}\sum_{k=1}^{n-1} x'_k(t)x'_{n-k}(t)dt + C_n\right]$$

$$= (t + C_0)\left[-\int \frac{1}{(t + C_0)^2}\sum_{k=1}^{n-1} x'_k(t)x'_{n-k}(t)dt + C_n\right].$$

From this formula, one obtains

$$x_2(t) = (t + C_0)\left[-\int \frac{1}{(t + C_0)^2}(x_1'(t))^2 dt + C_2\right] = (t + C_0)\left[-\int \frac{C_1^2}{(t + C_0)^2} dt + C_2\right]$$

$$= (t + C_0)\left(\frac{C_1^2}{t + C_0} + C_2\right) = C_2(t + C_0) + C_1^2,$$

$$x_3(t) = (t + C_0)\left[-\int \frac{1}{(t + C_0)^2} 2x_1'(t)x_2'(t)dt + C_3\right] = (t + C_0)\left[-\int \frac{2C_1C_2}{(t + C_0)^2} dt + C_3\right]$$

$$= (t + C_0)\left(\frac{2C_1C_2}{t + C_0} + C_3\right) = C_3(t + C_0) + 2C_1C_2.$$

We suppose that $x_j(t) = C_j(t + C_0) + \sum_{k=1}^{j-1} C_k C_{j-k}$, $j = 2, 3, \dots, n-1$. Then $x_j'(t) = C_j$, $j = 1, 2, \dots, n-1$, hence

$$x_n(t) = (t + C_0)\left[-\int \frac{1}{(t + C_0)^2} \sum_{k=1}^{n-1} C_k C_{n-k} dt + C_n\right]$$

$$= (t + C_0)\left[\frac{1}{t + C_0} \sum_{k=1}^{n-1} C_k C_{n-k} + C_n\right] = C_n(t + C_0) + \sum_{k=1}^{n-1} C_k C_{n-k}.$$

According to the mathematical induction axiom, the solution $x_n(t)$, $n = 2, 3, \dots$, are given by formula (9).

4.2. Proof of Theorem 2 by generating function method

We consider the generating function $G(t, z) = \sum_{n=0}^{\infty} x_n(t) z^n$ of the sequence of functions $x_n(t)$, defined by a formal series. By multiplying Equation (7) with z^n and summing, one obtains $\sum_{n=0}^{\infty} x_n(t) z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n}$ $x_k'(t) x_{n-k}'(t) z^n$. Considering the formula for the product of the power series, one obtains the differential equation $G(t, z) = \left[\frac{\partial}{\partial t} G(t, z)\right]^2$, which takes the form $\dfrac{\frac{\partial}{\partial t} G(t, z)}{\sqrt{G(t, z)}} = 1$. By integration, one obtains $2\sqrt{G(t, z)} = t + C(z)$, where $C(z) = C_0 + 2\sum_{n=0}^{\infty} C_n z^n$ is an arbitrary constant with respect to the variable t. Therefore, we have

$$G(t, z) = \frac{1}{4}[t + C(z)]^2 = \frac{1}{4}\left[t + C_0 + 2\sum_{n=1}^{\infty} C_n z^n\right]^2$$

$$= \frac{1}{4}(t + C_0)^2 + (t + C_0)\sum_{n=1}^{\infty} C_n z^n + \left(\sum_{n=1}^{\infty} C_n z^n\right)^2.$$

Using again the formula for the product of the power series, one obtains the following form for the generating function:

$$G(t, z) = \frac{1}{4}(t + C_0)^2 + C_1(t + C_0)z + (t + C_0)\sum_{n=2}^{\infty} C_n z^n + \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} C_k C_{n-k} z^n.$$

By identifying the coefficients in the two expressions of $G(t, z)$, one obtains the formulas (8) and (9).

4.3. Reciprocal proof for Theorem 2

If the functions $x_n(t)$ are given by formulas (8) and (9), then we have

$$\sum_{k=0}^{n} x'_k(t)x'_{n-k}(t) = 2x'_0(t)x'_n(t) + \sum_{k=1}^{n-1} x'_k(t)x'_{n-k}(t)$$

$$= C_n(t + C_0) + \sum_{k=1}^{n-1} C_k C_{n-k} = x_n(t),$$

therefore these functions satisfy the Equation (7).

5. Differential Recurrence Equations with Combinatorial Auto-Convolution

Theorem 3. *The sequence* $y_n(t)$ *of solutions of the differential recurrence equation with combinatorial auto-convolution*

$$y'_n(t) = \sum_{k=0}^{n} \binom{n}{k} y_k(t) y_{n-k}(t), \quad n = 0, 1, 2, \dots, \tag{10}$$

so that the functions $y_n(t)/n!$ *are in geometric progression, are given by the formula*

$$y_n(t) = \frac{(-1)^{n+1} n!\, C_1^n}{(t + C_0)^{n+1}}, \quad n = 0, 1, 2, \dots, \tag{11}$$

where C_0 *and* C_1 *are the constants from Theorem* 1.

Theorem 4. *The solutions of the equation*

$$y_n(t) = \sum_{k=0}^{n} \binom{n}{k} y'_k(t) y'_{n-k}(t), \quad n = 0, 1, 2, \dots, \tag{12}$$

are given by the formulas

$$y_0(t) = \frac{1}{4}(t + C_0)^2, \quad y_1(t) = C_1(t + C_0), \tag{13}$$

$$y_n(t) = n!\, C_n(t + C_0) + n! \sum_{n=1}^{n-1} C_k C_{n-k}, \quad n = 2, 3, \dots, \tag{14}$$

where C_n *are the constants from Theorem* 2.

Remark. (1) The Theorem 3, respectively 4, follows by Theorem 1, respectively 2, if we make the substitution $y_n(t) = n!\, x_n(t)$, $n = 0, 1, 2, \dots$.

(2) The differential recurrence equations with combinatorial auto-convolution (10) and (12) can also be solved directly using the *exponential*

generating function of the sequence of functions $(y_n(t))$, given by the formula $E(t, z) = \sum_{n=0}^{\infty} x_n(t) \frac{z^n}{n!} = G\left(t, \frac{z^n}{n!}\right)$.

6. Applications to Discrete Linear Time-Invariant Physical Systems Theory

A *discrete linear physical system is* a linear operator U which to every sequence of functions $i(t) = (i_n(t))$, named *input* of the system, makes to correspond a sequence of functions $o(t) = (o_n(t))$, named *output*, hence $U(i(t)) = o(t)$. If the system is *time-invariant*, namely, the operator U permutes with translations, hence $U((i_{n+k}(t))) = (o_{n+k}(t))$, $k = 1, 2, \ldots,$ then the system U has a sequence of functions $x(t) = (x_n(t))$, named *weighted function* or *impulse response function* of the system, such that the system has the discrete convolution form $o(t) = U(i(t)) = x(t) * i(t)$. If V is a second such physical system, having $y(t) = (y_n(t))$ as impulse response function, then the system $U \circ V$ obtained by the series connection of the two systems has the sequence $x(t) * y(t)$ as impulse response function, while the system $U + V$ obtained by the parallel connection of the two systems has the sequence $x(t) + y(t)$ as impulse response function. The above results are related to these topics. The Theorem 1 gives the impulse response $(x_n(t))$ of a system U which connected in series with itself gives a new system with the impulse response $(x'_n(t))$. The Theorem 2 gives the impulse response $(x_n(t))$ of a system U obtained by connecting in series with itself the system, which has the impulse response $(x'_n(t))$.

7. Conclusion

We presented here simple equations of a new type and some of their methods of solving. Because these equations have first order, their methods of solving are based on elementary mathematics, hence they can be used in the learning process as a complement to the simplest types of ordinary differential equations. We can consider numerous examples of differential recurrence equations with auto-convolution. For example, the equation considered in the Theorem 1 is only the first of the 96 equations given in [5] as examples of the general theory presented there. Besides these educational purposes, these new types of differential recurrence equations, which can be called *hybrid equations*, have applications not only in the physical systems theory, as stated in the previous section, but also in other areas as in seismology, geophysics, computer tomography, image processing, probabilities, statistics, queuing theory and more. The author hopes that this paper stimulates the solving of more complicated differential recurrence equations, possibly of higher order.

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